Comment on "Numerical methods for stochastic differential equations"

Kevin Burrage and Pamela Burrage
Advanced Computational Modelling Centre, University of Queensland, Brisbane QLD 4072, Australia

Desmond J. Higham
Department of Mathematics, University of Strathclyde, Glasgow G1 1XH, United Kingdom

Peter E. Kloeden
Fachbereich Mathematik, Johann Wolfgang Goethe Universität, D-60054 Frankfurt am Main, Germany

Eckhard Platen
School of Finance & Economics and Department of Mathematical Sciences, University of Technology Sydney, P.O. Box 123, Broadway NSW 2007, Australia

(Received 6 October 2005; revised manuscript received 11 May 2006; published 22 December 2006)

Wilkie [Phys. Rev. E **70**, 017701 (2004)] used a heuristic approach to derive Runge-Kutta-based numerical methods for stochastic differential equations based on methods used for solving ordinary differential equations. The aim was to follow solution paths with high order. We point out that this approach is invalid in the general case and does not lead to high order methods. We warn readers against the inappropriate use of deterministic calculus in a stochastic setting.

DOI: 10.1103/PhysRevE.74.068701 PACS number(s): 02.70.−c, 03.65.−w, 02.50.−r

Wilkie's paper [1] (and a follow-up paper [2]) recalls the confusion that plagued the scientific community during the 1960s and 1970s due to the misuse of the familiar deterministic calculus in stochastic modeling and for deriving numerical schemes for stochastic differential equations (SDEs). The claim that simple high order Runge-Kutta methods can be derived for general SDEs is incorrect and therefore misleading for researchers developing and applying numerical methods. In what follows, by default, we intend the words "order" and "convergence" to be interpreted in the classical *strong* (as opposed to weak) sense; see Ref. [3], [Chapters 10–13]. This is consistent with the pathwise arguments and computations in Ref. [1].

In 1982 Rümelin [4] looked at the possibility of extending Runge-Kutta schemes for deterministic ordinary differential equations in a natural manner to the SDE context, using only simple increments ΔW of the driving Wiener processes. There it was proved that such methods either do not converge when applied to Ito stochastic differential equations or they converge with at most a strong order 1.0. Earlier work by Clark and Cameron [5] also gave detailed arguments about the maximum rate of strong convergence when only Wiener increments are used. It is now well known in the numerical SDE community that one needs more information about the Wiener processes in each discretization subinterval, as given by multiple stochastic integrals, as well as the use of appropriate additional commutators ([6,7]), in order to obtain a higher order approximation; see, for example, Ref. [3] for details of how this follows from a unified theory of stochastic Taylor expansions.

Wilkie had the good fortune to derive convergent schemes. The explanation for this can be seen in Ref. [8], where it was shown that certain deterministic Runge-Kutta schemes adapted to Stratonovich SDEs converge to the solution of the corresponding Ito SDE obtained by an appropriate modification to the drift coefficient. Equation (7) in Wilkie's paper to which he applies the Runge-Kutta scheme (8) is precisely the modified Stratonovich SDE corresponding to the original Ito SDE.

The key equation used by Wilkie, namely,

$$dW_i dW_j = \delta_{ij}\Delta t,$$

is only correct in the mean and there is no justification for its use as an operator. Thus for general problems, all methods derived in this way will have weak order 1, although for certain special classes of problems the order can be higher.

Wilkie also claims (in Ref. [2]) that he can achieve higher order by using a variable step size to avoid situations in which the Wiener increments or derivatives are too large. But if a small enough constant stepsize is used then the orders (that Wilkie claims) should still be transparent. Furthermore, there are subtle issues of maintaining the Brownian path when using variable stepsize SDE implementations that do not allow for the rejection of random samples (see, for example, Ref. [9]).

Wilkie's discussion of convergence is at most only suggestive, and Wilkie in no way demonstrates the actual order of convergence of the proposed methods. Serious computational practice requires many thousands of sample paths to be simulated with log-log plots of RMS errors against step size, incorporating confidence intervals. Also, test problems must be chosen with care. Those SDEs (11) and (13) that Wilkie chose with multiple independent Wiener processes have a commutative noise structure (see Ref. [3], pages 348 ff.). Rümelin [4] showed the strong convergence order barrier is slightly higher for commutative noise than in the general case, specifically 1.5 rather than 1.0. Essentially the double stochastic integrals in a stochastic Taylor expansion of the solution of Wilkie's SDE (9) have the same coefficients, and the sum

$$\int_{t_n}^{t_{n+1}} \int_{t_n}^{t} dW_s^1 dW_t^2 + \int_{t_n}^{t_{n+1}} \int_{t_n}^{t} dW_s^2 dW_t^1$$

may be replaced by

$$\int_{t_n}^{t_{n+1}} dW_t^1 \int_{t_n}^{t_{n+1}} dW_t^2,$$

which is simply $\Delta W_n^1 \Delta W_n^2$. In addition, one should remember that convergence orders are the worst case over a class of SDEs, with a higher order possible for specific SDEs if certain terms happen to vanish. Similarly, a small coefficient accompanying the dominant Δt power may disguise the true order. Practical experience has also taught us that where numerical methods involve terms that are not relevant for achieving the desired order, these terms tend to cause numerical instabilities. If one follows Wilkie's approach, then one is likely to end up with methods that may suffer under this effect.

We want to emphasize that for some classes of simple problems, higher orders can be achieved, but this is not the case in general and that is why methods for SDEs are often considerably more sophisticated than in the ODE setting. Because these are general methods, they may be slower than methods designed with a special problem in mind. However, computational tools are now becoming available to simplify the implementation of stochastic numerical methods; for example MAPLE routines in Ref. [10,11] allow one to determine the coefficients of numerical methods for high dimensional systems and to check for structural simplifications such as commutative noise.

We finish with a general note of *warning* regarding the use of numerical methods to solve SDEs: forget numerical methods that have been derived for deterministic ordinary differential equations as Wilkie's approach will never lead to high order methods. Instead use numerical methods that have been designed specifically for SDEs such as the stochastic Taylor methods and the derivative-free stochastic Runge-Kutta methods in Ref. [3]. Then one will always be sure of not only convergence but convergence at a desired higher order (Ref. [12]).

We thank Roberto Vio for bringing Ref. [1] to our attention. We would also like to thank the anonymous referee who applied order 1, 2, 3, and 4 Runge-Kutta methods to a variational calculation of the ground state energy of the Helium atom with two electrons (the standard fourth-order scheme is cited in Wilkie's work). This problem is of dimension 6 with additive noise. All of these methods were observed to converge only with weak order 1 apart from the Heun method which coincides with a weak second order Runge-Kutta Langevin method derived in Ref. [12].

[1] J. Wilkie, Phys. Rev. E **70**, 017701 (2004).
[2] J. Wilkie and M. Cetinbas, Phys. Lett. A **337**, 166 (2005).
[3] Peter E. Kloeden and Eckhard Platen, *Numerical Solution of Stochastic Differential Equations* (Springer-Verlag, Berlin, 1992). Third corrected printing 1998.
[4] W. Rümelin, SIAM (Soc. Ind. Appl. Math.) J. Numer. Anal. **19**, 604 (1982).
[5] J. M. Clark and R. J. Cameron, in *Stochastic Differential Systems*, edited by B. Grigelionis, Volume 25 of Lecture Notes in Control and Information Sciences (Springer-Verlag, Berlin, 1980), pp. 162–171.
[6] K. Burrage and P. M. Burrage, Physica D **133**, 34 (1998).
[7] K. Burrage and P. M. Burrage, SIAM (Soc. Ind. Appl. Math.) J. Numer. Anal. **38**, 1626 (2000).
[8] P. E. Kloeden and R. Pearson, J. Aust. Math. Soc. Ser. B, Appl. Math. **20**, 8 (1977).
[9] J. G. Gaines and T. J. Lyons, SIAM J. Appl. Math. **57**, 1455 (1997).
[10] S. Cyganowski, L. Grüne, and P. E. Kloeden, in *Theory and Numerics of Differential Equations*, edited by J. F. Blowey, J. P. Coleman, and A. W. Craig (Springer-Verlag, Berlin, 2001), pp. 127–178.
[11] Desmond J. Higham and Peter E. Kloeden, in *Programming Languages and Systems in Computational Economics and Finance*, edited by Søren S. Nielsen (Kluwer, Boston, 2002), pp. 233–269.
[12] E. Helfand, Bell Syst. Tech. J. **58**, 2289 (1979).